Hamilton's Principle

Hamilton's Principle may be stated as follows.

" Of all the possible paths along which a dynamical system may move from one point to another within a given interval of time (consistent with constraints, if any) the actual path followed is that which minimises the time integral of the Lagrangean.

The principle can be alternatively be stated as:

The motion of the system from instant t_1 to instant t_2 is such that the line integral

$$J = \int_{t_1}^{t_2} L\,dt \qquad \text{——— (1)}$$

where $L = K.E - P.E = T - V$, is an extremum for the path of motion

In terms of the Calculus of variation, we can state Hamilton's principle as

$$\delta J = \delta \int_{t_1}^{t_2} L\,dt = 0 \qquad \text{——— (2)}$$

with variation zero at $t = t_1$ and $t = t_2$.

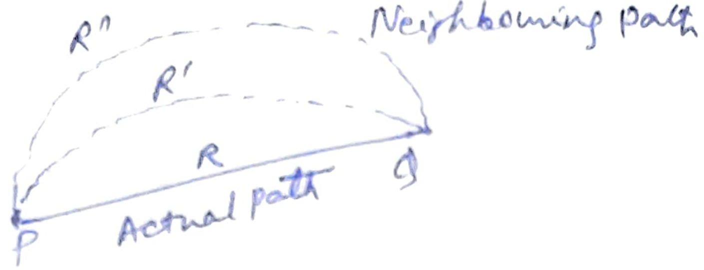
L can be expressed as

$$L = L(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \dot{q}_3 \ldots \dot{q}_n; t)$$

$$L = L(q_i, \dot{q}_i, t)$$

Therefore, eqn. (2) can be written as

$$\delta J = \delta \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt = 0 \qquad \text{——— (3)}$$



Consider the Conservative holonomic dynamical System from P to Q where P and Q are initial and final Configuration of the system at time t_1 and t_2 respectively. Let PRQ is the actual path and PR'Q, PR''Q, the two neighbouring paths out of infinite no. of possibilities.

For the deduction of Hamilton principle, the following two Conditions must be satisfied.

(i) δt must be equal to zero at end points i.e, at time t_1 the particle must be at P and at time t_2 the particle must be at Q.

(ii) δr must be equal to zero at end points i.e, the points P and Q are fixed in space.

Let the system be acted upon by a number of forces represented by F_i. Let ~~i~~ i^{th} particle of the system acted upon by force F_i acquire acceleration $\ddot{r}_i$.

So, that we write

$$F_i = m_i \ddot{r}_i$$

From D'Alembert's principle, we have

$$\sum_i \left(F_i - m_i \ddot{r}_i \right) \cdot \delta r_i = 0$$

$$\Rightarrow \sum_i F_i \delta r_i - \sum_i m_i \ddot{r}_i \delta r_i = 0 \quad \text{———} \quad (4)$$

But $\dfrac{d}{dt}(\dot{r}_i \delta r_i) = \dot{r}_i \dfrac{d}{dt}(\delta r_i) + \ddot{r}_i \delta r_i$

$$\Rightarrow \ddot{r}_i \delta r_i = \frac{d}{dt}(\dot{r}_i \delta r_i) - \dot{r}_i \frac{d}{dt}(\delta r_i) \quad \text{———} \quad (5)$$

If there is a finite variation along the actual and neighbouring paths.

$$\delta r_i = r_i' - r_i \quad (\text{say})$$

Then, $\frac{d}{dt}(\delta r_i) = \frac{d}{dt}(r_i' - r_i) = \frac{dr_i'}{dt} - \frac{dr_i}{dt} = \delta\left(\frac{dr_i}{dt}\right)$

$$= \delta(\dot{r}_i) \quad \text{--- (6)}$$

Here, primes has been used for neighbouring paths.

Putting $\ddot{r}_i \delta r_i$ from eqn. (5), to equation (4), we get

$$\sum_i F_i \delta r_i - \sum m_i \left[\frac{d}{dt}(\dot{r}_i \delta r_i) - \dot{r}_i \frac{d}{dt}(\delta r_i)\right] = 0$$

From (6), this may be written as

$$\sum_i F_i \delta r_i - \sum m_i \left[\frac{d}{dt}(\dot{r}_i \delta r_i) - \dot{r}_i \delta(\dot{r}_i)\right] = 0$$

$$\Rightarrow \sum_i F_i \delta r_i - \sum m_i \left[\frac{d}{dt}(\dot{r}_i \delta r_i) - \frac{1}{2}\delta(\dot{r}_i)^2\right] = 0$$

$$\Rightarrow \sum_i F_i \delta r_i + \sum_i \frac{1}{2} m_i \delta(\dot{r}_i)^2 = \sum \frac{d}{dt}(m_i \dot{r}_i \delta r_i)$$

$$\Rightarrow \sum_i F_i \delta r_i + \delta\left(\sum_i \frac{1}{2} m_i \dot{r}_i^2\right) = \sum \frac{d}{dt}(m_i \dot{r}_i \delta r_i) \quad \text{--- (7)}$$

But $\sum_i F_i \delta r_i =$ Work done by the force F_i during displacement

$$\sum F_i \delta r_i = \delta W \quad (\text{say})$$

and $\sum_i \frac{1}{2} m_i \dot{r}_i^2 =$ kinetic energy of the system $= T$

Therefore, equation (7) becomes.

$$\delta W + \delta T = \sum_i \frac{d}{dt}(m_i \dot{r}_i \delta r_i)$$

Integrating above equation between the limit t_1 and t_2, we get.

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt}(m_i \dot{r}_i \delta r_i)\, dt$$

$$= \sum_i \int_{t_1}^{t_2} d(m_i \dot{r}_i \delta r_i)$$

$$= \sum_i \left[m_i \dot{r}_i \delta r_i \right]_P^Q = 0, \text{ since } \delta r_i = 0$$

at the end points P and Q.

For a Conservative system. we know that

$\delta W = -\delta V$; where V is the potential energy

$$\Rightarrow \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} \delta (T - V)\, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

$$\Rightarrow \delta \int_{t_1}^{t_2} L\, dt = 0$$

or $\int_{t_1}^{t_2} L\, dt = J$ = extremum, which is Hamilton principle

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